ON A MODEL FOR RANK ANALYSIS IN FRACTIONAL PAIRED COMPARISONS

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Introduction

Statistical technology applied for the analysis of numerical data is appropriate only when the data conform the assumptions and the requirements needed in the statistical analysis. The mathematical model for fractional paired comparisons was postulated in such a way that these were mathematically workable and easy to apply and interpret. The objective of this paper is to develop a procedure for testing the appropriateness of the model and also to investigate the reliability of the estimators used in fractional paired comparisons.

2. Review of the model and tests of hypothesis

Various procedures for the analysis of paired comparisons are available. The method of analysis depends on the form in which the data are recorded. Measurements or scores may be avilable for different items under comparison or sometimes units in each pair may be ranked for acceptability.

In fractional paired comparisons, only those pairs are studied whichcontain a particular treatment (say T_1). Thus out of t treatments $T_1, \ldots T_t$, we shall take only $(T_1, T_2), (T_1, T_3), \ldots, (T_1, T_t)$ pairs for ranking. The total number of pairs in this case will be $(t-1)$. It is also postulated that with each of the treatments $T_1, \ldots T_t$, there exist parameters $\pi_1, \ldots \pi_t$, such that $\pi_i \geqslant 0$ and

$$\sum_{i=1}^{t} \pi_i = 1.$$

The behaviour of the parameters is further defined with a probability statement that

$$P(T_1 > T_i) = \frac{\pi_1}{\pi_1 + \pi_i} \qquad \ldots(1)$$

in the comparison of T_1 with T_i. Experimental observations are limited to ranking of items in pairs. We define r_{1ik} to be the ranks of T_1 when it is compared with T_i in the Kth replication of the design. Tied ranks are not permitted in the model and this makes r_{1ik} to take a value either one or two with $r_{1ik}+r_{i1k}=3$. The rank one is assigned to the treatment of a pair which is judged superior on the basis of the test attribute. If the design is repeated n times, the likelihood function is given by

$$L=\frac{\pi_1^{\,2n(t-1)-\sum_{i=2}^{t}\sum_{k=1}^{n} r_{1ik}} \prod_{i=2}^{t} \pi_i^{\,2n-\sum_{k=1}^{n} r_{i1k}}}{\prod_{i=2}^{t}(\pi_1+\pi_i)^n} \quad \ldots(2)$$

The maximum likelihood estimator of π_i is denoted *by* p_i. Procedures for obtaining these estimators are outlined by Rai and Sadasivan [5]. Here we shall require to refer to (2) for developing the theory for testing the appropriateness of the model. The test procedure is as follows :

Test (1) ; $H_o:\pi_i=1/t$ for all i against the alternative

$H_1 : \pi_i \neq \frac{1}{t}$ for some i

If we assume λ_1 to be likelihood ratio statistic for test (1) then

$$-2\log\lambda_1=2n(t-1)\log_e 2+2\Big[\Big\{2n\,(t-1)-\sum_{i=2}^{t}\sum_{k=1}^{n} r_{lik}\Big\}\log_e^{p_1}$$

$$+\sum_{i=2}^{t}(2n-\sum_{k=1}^{n} r_{i1k})\log_e p_i-\sum_{i=2}^{t} n\log_e(p_1+p_i)\Big] \quad \ldots(3)$$

has for large values of n, the x^2 distribution with $(t-1)$ degrees of freedom. This test is a test of treatment equality in the fractional paired comparisons.

3. Test for Appropriateness of the Model

In an experiment involving fractional paired comparisons, we compare treatments T_1 and T_i ($i=2, \ldots . t$) in pairs and obtain a parameter $\bar{\pi}_{1i}$, the probability that T_1 is ranked above T_i. The complementary probability will be $\bar{\pi}_{i1}=1-\bar{\pi}_{1i}$. Now if t treatments are involved in the experiment of this type, one has to estimate (t—1) parameters. These estimates are obtained from the relative frequencies. These are given by

$$\bar{p}_{1i}=f_{1i}/n \quad \ldots(4)$$

where $\bar{p}_{1i}$ is the estimator of $\bar{\pi}_{1i}$, f_{1i} is the number of times T_1 is rated above T_i and n is the number of replications. The likelihood function may be written as

$$\bar{L}(\bar{\pi}_{1i}) = \prod_{i} \bar{\pi}_{1i}^{\,f_{1i}}\,\bar{\pi}_{i1}^{\,f_{i1}} \quad \ldots(5)$$

where $f_{li}+f_{il}=n$ and $\bar{\pi}_{li}+\bar{\pi}_{il}=1$. The relations between the frequencies and rank sums are as given below :

$$f_{li}=2n(t-1)-\sum_{k=1}^{n} r_{lik} \text{ and}$$

$$f_{il}=2n-\sum_{k=1}^{n} r_{ilk} \text{ for } i=2, \ldots t.$$

In order to test the appropriateness of the model of fractional paired comparisons, the following test is proposed. Take the null hypothesis,

$$H_o : \bar{\pi}_{li}=\frac{\pi_l}{\pi_1+\pi_i} \text{ for all } i \text{ against}$$

$$H_1 : \bar{\pi}_{1i}\neq\frac{\pi_l}{\pi_l+\pi_i} \text{ for some } i.$$

The likelihood ratio depends on $\bar{L}\,(\bar{p}_{li}/H_o)$ and $\bar{L}\,(\bar{p}_{li}/H_1)$ where $\bar{L}$ is defined by (5) and these functions represent evaluations of $\bar{L}$ in terms of estimators $\bar{p}_{li}$ obtained under the hypotheses H_o and H_1 respectively. Again it may be seen that $\bar{L}\,(\bar{p}_{li}/H_o)=L$ defined in (2). These quantities may be evaluated using the maximum likelihood estimators and we have the following relations.

$$\log \bar{L}\,(\bar{p}_{li}/H_o)=\sum_{i=2}^{t} n \log_e(p_1+p_i)-\left\{2n\,(t-1)\sum_{i=2}^{t}\sum_{k=1}^{n} r_{lik}\right\}\log_e p_1$$

$$-\sum_{i=2}^{t}\,(2n-\sum_{k=1}^{n} r_{ilk})\log_e p_i \qquad \ldots(6)$$

and

$$\log_e \bar{L}\,(\bar{p}_{li}/H_1)=\Sigma f_{li}\log_e f_{li}-n(t-1)\log_e n \qquad \ldots(7)$$

For testing the appropriateness of the model, the statistic

$$-2\log_e \lambda_2=2\,[\Sigma f_{li}\log_e f_{li}-n(t-1)\log_e n-\{2n(t-1)$$

$$-\sum_{i=2}^{t}\sum_{k=1}^{n} r_{lik}\}\log_e p_l-\sum_{i=2}^{t}\,(2n-\sum_{k=1}^{n} r_{ilk})\log_e p_i+$$

$$\sum_{i=2}^{t} n\log_e (p_1+p_i) \qquad \ldots(8)$$

is calculated which follows the x^2 distribution with $(t—l)$ degrees of freedom for large values of n.

In test of goodness of fit, the procedures involved require the expected cell frequencies, say E and computing x^2 by taking sums of terms of the form $(O-E)^2/E$ where O is the observed frequencies. Expected cell frequencies are related to the estimator $p_1, \ldots p_t$. In case of fractional paired comparisons, the expected frequencies are given by

$$f'_{1i}=np_1(p_1+p_i)\ ;\ i=2, \ldots . t \ldots \qquad \ldots(9)$$

In terms of observed and expected frequencies the expression (8) may be written as

$$-2 \log_e \lambda_2 = 2\Sigma_i f_{1i} \log_e (f_{1i}/f'_{1i}) \qquad \ldots(10)$$

We shall now simplify (10) by taking $f_{1i}/f'_{1i}=1+e_{1i}$ where e_{1i} is either positive or negative. We now have

$$-2 \log_e \lambda_2 = 2\Sigma f'_{1i}(1+e_{1i}) \log_e (1+e_{1i})$$

We shall now use the power series expansions of the logarithms and stop with the second term. The errors in doing so will not be large if $| e_{1i} |$ is small. Expanding the logarithmic series and simplifying the expression, we have

$$-2 \log_e \lambda_2 \approx \Sigma_i (f_{1i}-f'_{1i})^2/f'_{1i} \qquad \ldots(11)$$

This is the usual form of goodness of fit test.

4. Asymptotic Distribution of the Estimator of π_i

Let us define X_i as the number of times treatment i obtains a rank 1 in fractional paired comparisons. The likelihood function (2) in terms of X_i is given by

$$L= \frac{\pi_1^{X_1} \prod_{i=2}^{t} \pi_i^{X_i}}{\prod_{i=2}^{t} (\pi_1+\pi_i)^n} \qquad \ldots(12)$$

We may define for convenience that

$$\lambda_{ii}= \frac{1}{\pi_i} \Sigma\ \pi_j(\pi_i+\pi_j)^{-2}\ ;\ \text{for } i=1,\ldots t \ldots \qquad \ldots(13)$$

and

$$\lambda_{ij}=-(\pi_i+\pi_j)^{-2} \text{ for } i\neq j\ ;\ i,j=1, \ldots t. \ldots \qquad \ldots(14)$$

Let x_1 be an indicator variate with the value unity if treatment T_1 ranks above T_j and zero otherwise. Similarly x_{j1} has the value 1 when T_j ranks above T_1 and zero otherwise.

Then

$$X_1 = \sum_{j=2}^{t} x_{lj} \text{ and } x_j=0 \text{ or } 1 \text{ for } j=2, \ldots . t.$$

Here x_{lj} is a binomial variate with expectation $\pi_l(\pi_l+\pi_j)^{-1}$ and variance $\pi_l\pi_j$ $(\pi_l+\pi_j)^{-2}$. The variates x_{lj} making up the sum x_l, are independent in probability and it follows that

$$E(X_l)=\pi_l \sum_{j=2}^{t} (\pi_l+\pi_j)^{-1} \qquad \ldots(15)$$

$$V(X_l)=\pi_l \sum_{j=2}^{t} \pi_j(\pi_l+\pi_j)^{-2} \qquad \ldots(16)$$

and

$$\text{cov. } (X_l, X_j)=-\pi_l\,\pi_j\,(\pi_l+\pi_j)^{-2}\,;\,(j=2, \ldots t) \qquad \ldots(17)$$

The parameters $\pi_l, \ldots, \pi_t$ are subjected to the restriction that

$$\sum_{i=1}^{t} p_i=1.$$

Therefore we may regard $p_l, \ldots, p_{t-l}$ as maximum likelihood estimators of the independent parameters $\pi_l, \ldots \pi_{l-l}$; taking

$$\pi_t=1-\sum_{i=1}^{t-1} p_i.$$

For large values of n, $\sqrt{n}\,(p_l-\pi_l), \ldots, \sqrt{n}\,(p_{t-l}-\pi_{t-l})$ have the multivariate normal distributions with zero means and dispersion matrix given by

$$[\lambda'_{ij}]^{-1} \text{ where } \lambda'_{ij}=\lambda_{ij}-\lambda_{it}-\lambda_{jt}+\lambda_{tt} \qquad \ldots(18)$$

If σ_{ij} is the covariance of $\sqrt{n}\,(p_i-\pi_i)$ and

$\sqrt{n}\,(p_j-\pi_j)$ for $i, j=1, \ldots (t-1)$, then

$$\sigma_{ij}=\frac{\text{cofactor of } \lambda_{ij} \text{ in } \begin{vmatrix} [\lambda_{ij}] & [I]' \\ [\,I\,] & o \end{vmatrix}}{\begin{vmatrix} [\lambda_{ij}] & [I]' \\ [\,I\,] & o \end{vmatrix}} \qquad \ldots(19)$$

where [I] and [I]′ are respectively row and column vectors of t unit elements. The remaining variance and covariances associated with $\sqrt{n}\,(p_t-\pi_t)$ may be obtained from the relationship

$$\sqrt{n}\,(p_t-\pi_t)=-\sum_{i=1}^{t-1}\sqrt{n}\,(p_i-\pi_i) \qquad \ldots(20)$$

so that

$$\sigma_{tt} = \sum_{i=1}^{t} \sum_{j=1}^{t-1} \sigma_{ij} \quad \ldots(21)$$

and

$$\sigma_{it} = -\sum_{j=1}^{t=1} \sigma_{ij}\ ;\ (i=1, \ldots (t-1) \quad \ldots(22)$$

5. An Illustrative Example

We shall use the data from a taste-testing experiment on the quality of chapaties prepared from different improved varieties of wheat as given by Sadasivan, Rai and Austin [6]. The chapaties were prepared from four varieties namely Sharbati sonora, Sonalika, K-65 and C-306 and the results of one of the judges gave the sums of ranks 19, 5, 6, 6 respectively for the chapaties of the different varieties for four replications. In the experiment only those pairs were studied where the variety Sharbati Sonora was present. The use of the table in that paper showed the following values.

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	Prob.
19	5	6	6	.17	.50	.17	.17	.1054

This result is not significant at 5% level of significance and is not indicative of any real difference in the taste quality of the chapaties of different varieties.

The different observed frequencies are obtained by the relations given in section 3 and are presented below :

$f_{12}=1$	(1.0149)	$f_{21}=3$	(2.9851)
$f_{13}=2$	(2.0000)	$f_{31}=2$	(2.0000)
$f_{14}=2$	(2.0000)	$f_{41}=2$	(2.0000)

The expected frequencies are also given in brackets corresponding to different observed frequencies. The value of $-2 \log_e \lambda_2$ from the expression (8) works out to be .0004 and the corresponding value from expression (11) is .0003. These values are very close to each other and they are taken to be values from x^2 distribution at 3 degrees of freedom. This result indicates that the proposed model for fractional paired comparisons is quite satisfactory for the data of this experiment.

We now obtain the estimates of variances and covariances of $\sqrt{np_1}, \ldots,$ $\sqrt{np_4}$ or $\sqrt{n}\,(p_1-\pi_1), \ldots, \sqrt{n}\,(p_4-\pi_4)$. In the first step we obtain the values of $\hat{\lambda}_{ij}$ from substitution of values of p_i for π_i in (13) and (14). Then we have

$\hat{\lambda}_{11} = 23.82$ $\qquad$ $\hat{\lambda}_{23} = -2.23$

$\hat{\lambda}_{12} = 2.23$ $\qquad$ $\hat{\lambda}_{24} = -2.23$

$\hat{\lambda}_{13} = -8.65$ $\qquad$ $\hat{\lambda}_{33} = 23.82$

$\hat{\lambda}_{14} = -8.65$ $\qquad$ $\hat{\lambda}_{34} = -8.65$

$\hat{\lambda}_{22} = 6.66$ $\qquad$ $\hat{\lambda}_{44} = 23.82$

The estimate of the determinant in the denominator of (19) is

$$\begin{vmatrix} 23.82 & -2.23 & -8.65 & -8.65 & 1 \\ -2.23 & 6.66 & -2.23 & -2.23 & 1 \\ -8.65 & -2.23 & 23.82 & -8.65 & 1 \\ -8.65 & -2.23 & -8.65 & 23.82 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = -42045.48$$

Now for example from (19), we have

$$\hat{\sigma}_{12} = \frac{1}{42045.48} \begin{vmatrix} -2.23 & -2.23 & -2.23 & 1 \\ -8.65 & 23.82 & -8.65 & 1 \\ -8.65 & -8.65 & 23.82 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

$$= -0.0251$$

and similarly the complete set of estimated variances and covariances is

$\hat{\sigma}_{11} = 0.0289$ $\quad$ $\hat{\sigma}_{12} = -0.0251$ $\quad$ $\hat{\sigma}_{13} = -0.0019$ $\quad$ $\hat{\sigma}_{14} = -0.0019$

$\hat{\sigma}_{22} = 0.0752$ $\quad$ $\hat{\sigma}_{23} = -0.0251$ $\quad$ $\hat{\sigma}_{24} = -0.0251$ $\quad$ $\hat{\sigma}_{33} = 0.0289$

$\hat{\sigma}_{34} = -0.0019$ $\quad$ $\hat{\sigma}_{44} = 0.0289.$

The estimated variances and covariances of p_1, p_2, p_3 and p_4 may be obtained by dividing the above values by $n=4$. Consequently the estimated standard errors of p_1, p_2, p_3 and p_4 are respectively given by 0.085, 0.137, 0.085 and 0.085.

6. Discussion

The results obtained in this paper are valid for large samples. When the number of replications are few, the results are approximately correct. A test for testing the appropriateness of the model of fractional paired comparisons has been developed. This test is valid for the hypotheses which are used to test the equality of the treatment main effects. The test can be extended to cover the case of repetition of design in different groups or by different judges at different times. This model for fractional paired comparisons is found quite satisfactory in a number of taste-testing experiments conducted in Indian Agricultural Research Institute, New Delhi. The estimates of variances and covariances of the parameters $p_1, \ldots p_t$ are obtained. In extreme cases, the estimators $p_1, \ldots p_t$ have the set of values 1, 0,...0.

This presents difficulty in computing $\hat{\lambda}_{ij}$ and in estimating the variances and covariances. In this situation it is suggested to eliminate the treatment for which $p_i = 1$ and then obtain the estimates of the remaining parameters. In that case it will be possible to estimate the variances and covariances for the remaining parameters.

7. Summary

In this paper we have examined some of the properties of the method of fractional paired eomparisons. The results are asymptotically correct for large sample sizes. A test procedure for testing the appropriateness of the model for fractional paired comparisons has been developed. The test statistic is distributed as x^2 for large values of n and it has been transformed into the usual form of goodness of fit test. Formulae for variances and covariances of the estimates of treatment ratings have been obtained.

A practical example is discussed for judging the suitability of the model of fractional paired comparisons. Estimated variances and covariances of the estimators of the treatment ratings have been worked out.

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